Topological transitivity and recurrence as a source of chaos

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Abstract. This paper builds on previous work due to Glasner-Weiss [7], concerning the connection between topological transitivity, recurrence and sensitive dependence on initial conditions.

The basic feature of the phenomenon of deterministic chaos is the sensitive dependence on initial conditions, a property in oposition to stability. We shall discuss it in the context of topological dynamical systems, acting on metric spaces. In what follows M will denote a metric space and $\mathcal{S}$ will be one of the semigroups $\mathbf{N}, \mathbf{Z}, \mathbf{R}_+$, or $\mathbf{R}$.

Definition 1. (*J. Guckenheimer* [8]) *A topological dynamical system* $\Phi : \mathcal{S} \times M \to M$ *shows sensitive dependence on initial conditions* (*equivalently,* Φ *is sensitive*) *if there exists a* $\delta > 0$ *such that for every* $x \in M$ *and every neighbourhood* V *of* x *one can find a point* $y \in V$ *and a number* $t \in \mathcal{S},\ t > 0$, *for which*

$$d(\Phi_t x, \Phi_t y) > \delta.$$

The terminology above extends to the case of continuous mappings $T : M \to M$, by refering to the status of the discrete dynamical systems $(T^n)_{n \in \mathbf{N}}$ they generate. As well known, the behaviour of many dynamical systems can be settled by observing appropriate mappings.

The simplest example of a mapping which shows sensitive dependence on initial conditions is that of doubling angles on the unit circle,

$$T : S^1 \to S^1,\ T(z) = z^2.$$

In many cases (included this one) the sensitive dependence on initial conditions is a result of topological transitivity and abundance of nice orbits. To detail this assertion, we need a preparation, mostly due to E. Glasner and B. Weiss [7].

Given a topological dynamical system $\Phi : \mathcal{S} \times M \to M$, a point a of M is said to be *transitive* if its $\omega-$limit set is M. The systems Φ which admit transitive points are themselves called *topologically transitive*. Topological transitivity indicates the

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existence of complicated (dense) orbits. It could lead eventually to sensitivity, as it is the case where M is an interval. See [3].

If the phase space M has an isolated point, then all topological dynamical system acting on it are nonsensitive; just negate the definition of sensitivity. So, only the perfect metric spaces enter the story and in that case the restriction of the given system to the closure of each orbit is topologically transitive.

The following result says that topological transitivity is a stroboscopic property:

Lemma 1. (*H. Onishi; cf.* [1], *page 104*). *Let $(t_n)_n$ be a real sequence tending to ∞. Then there exists a residual subset A of $\boldsymbol{R}_+$ with the following property: for each $t \in A$ one can find a subsequence $(t_{k(n)})_n$ of $(t_n)_n$ and a sequence $(m_{k(n)})_n$ of natural numbers such that $t_{k(n)} - t \cdot m_{k(n)} \to 0$ as $n \to \infty$.*

The basic fact relating topological transitivity and chaotic behaviour is stated as follows:

Lemma 2. *Let $\Phi : \mathcal{S} \times M \to M$ be a topologically transitive nonsensitive dynamical system. Then for every $\varepsilon > 0$ there exist a transitive point $a \in M$ and a neighbourhood U of it such that*

$$\sup_{x\in U} \sup_{t\in\mathcal{S}_+} d(\Phi_t x, \Phi_t a) \leq \varepsilon.$$

Proof. Because Φ is nonsensitive, there must exist a point z and an open neighbourhood V of it such that

$$\sup_{y\in V} \sup_{t\in\mathcal{S}_+} d(\Phi_t z, \Phi_t y) < \varepsilon/2.$$

Letting b a transitive point of Φ, it follows that $a = \Phi_s b \in V$ for some $s \in \mathcal{S}_+$ and thus $U = B_\delta(a) \subset V$ for some $\delta > 0$. Clearly, a is also a transitive point for Φ and for every $x \in U$ and every $t \in \mathcal{S}_+$ we have

$$d(\Phi_t a, \Phi_t x) \leq d(\Phi_t a, \Phi_t z) + d(\Phi_t z, \Phi_t x) < \varepsilon. \blacksquare$$

Corollary 3. *Let $\Phi : \mathcal{S} \times M \to M$ be a topologically transitive nonsensitive dynamical system. Then there exist a $t > 0$ in $\mathcal{S}$ and a strictly increasing sequence $(k(n))_n$ of natural numbers such that*

$$\Phi_{k(n)\cdot t} \to id_M$$

uniformly as $n \to \infty$. Moreover, in the continuous time case, the set of all such t is a residual.

Proof. Notice first the existence of a $t > 0$ *in* $\mathcal{S}$ such that Φ_t is topologically transitive; see Lemma 1, for the continuous time case. Because Φ is nonsensitive, Φ_t is nonsensitive too. According to Lemma 2, for each $n \in \mathbf{N}^\star$ there exists a transitive point a_n and a neighbourhood U_n of a_n such that

$$\sup_{k\in\mathbf{N}} \sup_{x\in U_n} d(\Phi_{kt}x, \Phi_{kt}a_n) < 1/n$$

which yields, for each n, a $k(n) \in \mathcal{S}_+$, $k(n) \geq n$, such that $\Phi_{k(n)\cdot t}a_n \in U_n$. Then

$$\sup_{m\in\mathbf{N}} d(\Phi_{k(n)\cdot t}(\Phi_{m\cdot t}a_n), \Phi_{m\cdot t}a_n) < 1/n$$

which leads to

$$d(\Phi_{k(n)\cdot t}x, x) < 1/n$$

because of the transitivity of a_n. Consequently, $\Phi_{k(n)\cdot t} \to id_M$ uniformly, as $n \to \infty$. ■

The property outlined in Corollary 1 above reflects a certain kind of rigidity of the nonchaotic systems. The trajectories of the different points visit at the same moment of time all $\varepsilon-$ neighbourhoods, so the different patterns in the phase space tend to be recovered during the process of iteration.

Quite natural, the transitivity and the frequency at which the neighbourhoods are visited play a role in relating chaotic situations. We shall need the following definition introducing a class of fast recurrent points:

Definition 2. *A point* a *of* M *is said to be algebraically recurrent* (*for a topological dynamical system* $\Phi : \mathcal{S} \times M \to M$) *if for every neighbourhood* U *of it there exists a sequence* $(k(n))_n$ *of elements of* $\mathcal{S}_+$ *such that* $\Phi_{k(n)}a \in U$ *for every* $n \in \mathbf{N}$, $k(n) \to \infty$ *and one of the difference sets*

$$\begin{aligned} A_0 &= \{k(n) \,|\, n \in \mathbf{N}\} \\ A_n &= \{s - t \,|\, \textit{for all } t < s \textit{ in } A_{n-1}\},\ n \geq 1 \end{aligned}$$

has bounded gaps (*i.e., for a suitable* $L > 0$, *every interval* $[\alpha, \beta] \subset \mathbf{R}_+$ *with* $\beta - \alpha > L$ *contains an element in that set*).

Roughly speaking, algebraic recurrence means that each neighbourhood (of the point under attention) is visited at polinomial frequency. There are two particular cases, already noticed in the literature: In the case of uniformly recurrent (equivalently, almost periodic) points, A_0 has bounded gaps. In the case of regular points (i.e., the generic points a for which there exist invariant probability measures μ such that $\mu(U) > 0$ for every neighbourhood U of a), A_1 has bounded gaps; see [6], page 75, for details.

If a point a is algebraically recurrent, then all points of its orbit are of the same type. Consequently, we can speak of *algebraically recurrent orbits.*

Theorem 4. *Suppose that $\Phi : \mathcal{S} \times M \to M$ is a topological dynamical system satisfying the following two conditions:*

(T) *Φ is topologically transitive;*

(AR) *The union of all algebraically recurrent orbits is dense.*

Then either Φ shows sensitive dependence on initial conditions or Φ is nonsensitive and minimal.

Proof. Suppose that Φ is nonsensitive. We shall show that every point z of M is topologically transitive. For, we have to remark the following property of equicontinuity played by Φ :

For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d(y, z) < \delta \text{ implies } \sup_{t \in \mathcal{S}_+} d(\Phi_t y, \Phi_t z) < \varepsilon. \tag{1}$$

In fact, given $\varepsilon > 0$ we can choose (via Lemma 2) a transitive point a and a neighbourhood U of a such that

$$\sup_{x \in U} \sup_{t \in \mathcal{S}_+} d(\Phi_t x, \Phi_t a) < \varepsilon.$$

By (AR), there exists also an algebraically recurrent point $p \in U$ and thus a strictly increasing sequence $(k(n))_n$ of elements of $\mathcal{S}_+$, with the properties stated in Definition 2 above. By Lemma 1, we can assume also that all the mappings $\Phi_{k(n)}$ are topologically transitive (and thus they have dense images) ; for, replace the sequence $(k(n))_n$ by a translate of it, if necessary. Then for every $t \in \mathcal{S}_+$ and every $n \in \mathbf{N}$ we have

$$d(\Phi_{k(n)}\Phi_t a, \Phi_t a) \le d(\Phi_{k(n)+t} a, \Phi_{k(n)+t} p) + d(\Phi_{k(n)+t} p, \Phi_t a) < 2\varepsilon$$

and thus

$$\sup_{n \in \mathbf{N}} \sup_{x \in M} d(\Phi_{k(n)} x, x) \le 2\varepsilon$$

because the positive orbit of a is dense.

Let $0 \le m \le n$ in $\mathbf{N}$. Then

$$d(\Phi_{k(n)-k(m)}\Phi_{k(m)} x, \Phi_{k(m)} x) = d(\Phi_{k(n)} x, \Phi_{k(m)} x) \le 4\varepsilon$$

for every $x \in M$. Because the mappings $\Phi_{k(n)}$ have dense images, we get

$$\sup_{x\in M} d(\Phi_s x, x) \le 4\varepsilon$$

for every s in the difference set $A_1 = \{k(n) - k(m) \,|\, 0 \le m \le n \text{ in } \mathbf{N}\}$. Letting

$$\begin{array}{rcl} A_0 & = & \{k(n) \,|\, n \in \mathbf{N}\} \\ A_j & = & \{s - t \,|\, 0 \le t \le s \text{ in } A_{j-1}\},\ j \ge 1 \end{array}$$

it is clear that

$$\sup_{x\in M} d(\Phi_s x, x) \le 2^{j+1}\varepsilon \tag{2}$$

for every $s \in A_j$. Since p is uniformly recurrent, one of the above sets, say A_N has bounded gaps. Letting A_N as an increasing sequence $\{s(n) \,|\, n \in \mathbf{N}\}$, there exists a constant $L > 0$ such that $s(n+1) - s(n) \le L$ for every $n \in \mathbf{N}$.

A simple compactness argument shows that the family $(\Phi_t)_{t\in[0,L]}$ is equicontinuous at z, so to derive (1) from (2) (with $n = N$) it sufices to notice the equality

$$d(\Phi_t y, \Phi_t z) = d(\Phi_{t-s(n)}\Phi_{s(n)} y, \Phi_{t-s(n)}\Phi_{s(n)} z)$$

which works for all $t \in [s(n), s(n) + L]$.

Now, to conclude the proof of the transitivity of z, let $x \in M$ and $\varepsilon > 0$. Then choose a $\delta > 0$ as in (1) and remark that by transitivity of a there exists a pair t', t'' in $\mathcal{S}$, with $t'' > t' + 1/\varepsilon > t' > 0$, such that

$$d(\Phi_{t'} a, z) < \delta \text{ and } d(\Phi_{t''} a, x) < \varepsilon.$$

Then $t'' - t' > 1/\varepsilon$ and

$$d(\Phi_{t''-t'} z, x) \le d(\Phi_{t''-t'} z, \Phi_{t''-t'}\Phi_{t'} a) + d(\Phi_{t''} a, x) < 2\varepsilon,$$

which assures that $\omega(z) = M$. ■

If a system is minimal, then all points of its state space are uniformly recurrent. The case of irrational rotations shows that minimality alone is not strong enough to imply sensitivity. On the other hand, there exist sensitive systems which do not satisfy either (T) or (AR).

Example 1. (*Sensitivity without recurrence and topological transitivity*). *Consider the mapping*

$$T(x) = \begin{cases} 3x/2 & \text{if } \ 0 \le x \le 2/3 \\ -3x/2 + 2 & \text{if } \ 2/3 \le x \le 1. \end{cases}$$

Clearly, T is expansive and thus chaotic. Because the $\omega-$ limit set of $(0,1)$ is included in $[1/2, 1]$, no points in $(0, 1/2)$ is recurrent. Because the interval $(1/2, 1)$ is positively invariant, T cannot be topologically transitive.

R. Devaney [4] made the first attempt to define the term of chaotic dynamical system. Except for a redundancy noticed by J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey [2], his definition is as follows:

Definition 3. *A topological dynamical system Φ is said to be* Devaney chaotic *if it is topologically transitive, nonminimal and the union of all periodic orbits is dense.*

In the case of Devaney chaotic behaviour, the nonminimality is equivalent to the fact that the phase space contains an infinity of points. Devaney's original definition asked also for sensitivity, but it was noticed in [2] that the sensitivity follows from the other hypotheses. Of course, Theorem 1 covers the matter, but it is larger than the situation of Definition 3.

Theorem 1 has counterparts for attractors (i.e., bounded, closed, invariant and attractive sets). See Haraux [9], for details. We notice here only a particular case, strong enough to explain why chaotic behaviour is merely a common phenomenon than an exotic one.

Theorem 5. *Let Φ be a topological dynamical system acting on a metric space. Suppose that Φ has a compact attractor A such that:*

(P) *The set of periodic orbits of Φ is dense in A.*

(T) *$\Phi|A$ is topologically transitive.*

Then either A is a periodic attractor, or A is a strange attractor (i.e., the dynamics on it is sensitive).

The conditions (P) and (T) in Theorem 2 above are fulfilled for example by the so called *Axiom A attractors*. See [11], for details. Due to their hyperbolic structure, all these attractors show sensitive dependence on initial conditions. Particularly, this is the case of the horseshoe, the solenoid, Anosov's tori etc.

The conditions (T) and (AR) both passes to quotients, so in principle we can exhibit new examples of sensitive dynamical systems (or of strange attractors) by passing to appropriate quotients.

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